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## ON THE ECCENTRICITY OF PLANE SECTIONS OF QUADRICS.

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### I.

### CONJUGATE ECCENTRICITY OF A CONIC.

The foci of a conic, defined as the points of intersection of tangents to the conic from the two imaginary circular points at infinity, are four in number, two of them real and two imaginary. Salmon in his Conic Sections, chapter on Invariants and Covariants, gives a process of finding the co-ordinates of the foci of a conic from the equation. This method applied to the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, gives for the co-ordinates of the foci $(c, 0)$, $(-c, 0)$, $(0, ic)$, $(0, -ic)$, where $c = \sqrt{a^2 - b^2}$, and $i = \sqrt{-1}$. Thus it will be seen that the two real foci are situated on the transverse axis of the conic at equal distances from the centre, while the two imaginary foci consist of a pair of conjugate imaginary points situated on the conjugate axis, also equidistant from the centre.

Starting from this basis of fact, imaginary focal properties of conics may be developed which are counterparts of their real focal properties.* The method of development is parallel to that used by Salmon in proving the real focal properties.†

I shall call $e_1$ the conjugate eccentricity and proceed to find its relation to the ordinary eccentricity. Eliminating $b^2$ in the equations

$$e_1^2 = \frac{b^2 - a^2}{b^2} \quad \text{and} \quad e^2 = \frac{a^2 - b^2}{a^2},$$

and dividing by $a^2$, we have

$$e_1^2 = \frac{e^2}{e^2 - 1}; \quad \text{similarly,} \quad e^2 = \frac{e_1^2}{e_1^2 - 1}.$$

* [In fact it may be readily seen that, in problems relating to conics, the minor axis may be treated as though it were the major axis by merely interchanging $a$ and $b$, and $x$ and $y$. In this case $e$ becomes $e_1$.—*O. S.*]

† [The details are omitted, since the process suggested in the preceding note is much simpler.—*O. S.*]

When $e_1 = 0$, the curve is a circle; when $e_1$ is imaginary, the curve is an ellipse; when $e_1 = \infty$, the curve is a parabola; when $e_1$ is real and greater than unity, the curve is an hyperbola; when $e_1$ is real and less than unity, the curve is such that the ordinary eccentricity is imaginary.

## II.

### TO FIND THE ECCENTRICITY OF ANY PLANE SECTION OF A QUADRIC SURFACE.

Let the triaxial ellipsoid, whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, be the type of the most general surface. By giving to the constants the proper signs and values we may obtain all the results wished.

Since the sections of a quadric by parallel planes are similar conics, we need to consider only central sections of the surface. The lengths of the axes of any central section of a quadric are given in Salmon's Geometry of Three Dimensions, Art. 101 (3d ed.), as follows: Let $a'$ and $b'$ be the semiaxes of the section, $a$, $b$, and $c$ the semiaxes of the quadric, and $\alpha$, $\beta$, $\gamma$ the direction angles of the normal to the plane of section. We have then

$$\frac{1}{a'^2} + \frac{1}{b'^2} = \frac{\sin^2\alpha}{a^2} + \frac{\sin^2\beta}{b^2} + \frac{\sin^2\gamma}{c^2} = S. \tag{1}$$

$$\frac{1}{a'^2 b'^2} = \frac{\cos^2\alpha}{b^2 c^2} + \frac{\cos^2\beta}{a^2 c^2} + \frac{\cos^2\gamma}{a^2 b^2} = C. \tag{2}$$

Eliminating $b'^2$ in these equations and solving for $a'^2$, we find

$$a'^2 = \frac{S \pm \sqrt{S^2 - 4C}}{2C}. \tag{3}$$

Eliminating $a'$ and solving for $b'$ we have identically the same result. Hence we must take the upper sign before the radical for the one and the lower sign for the other. The upper sign will of course give the longer axis. Substituting these values in the formula, $\varepsilon^2 = 1 - \frac{b'^2}{a'^2}$, we have

$$\varepsilon^2 = \frac{2\sqrt{S^2 - 4C}}{S + \sqrt{S^2 - 4C}}. \tag{4}$$

In order to simplify this expression it is necessary to replace $S$ and $C$ by their values and reduce the quantity under the radical to the form of a perfect square. But this is generally impossible.

There are, however, some particular cases where the reduction is possible. Thus let one of the angles, say $\alpha$, be zero; then, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\cos^2\beta$ and $\cos^2\gamma$ are each equal to zero. In such a case the section is a principal section and its eccentricity is already known.

Again, if one of the angles, say $\beta$, is $90°$, then $\cos^2\alpha = \sin^2\gamma$ and $\sin^2\alpha = \cos^2\gamma$; whence we may write

$$S = \frac{\sin^2\alpha}{a^2} + \frac{1}{b^2} + \frac{\cos^2\alpha}{c^2}, \tag{5}$$

and

$$\sqrt{(S^2 - 4C)} = \pm\left(\frac{\sin^2\alpha}{a^2} - \frac{1}{b^2} + \frac{\cos^2\alpha}{c^2}\right). \tag{6}$$

Substituting in (4), employing the lower sign of the right hand member of (6), and writing $\varepsilon_2$ for $\varepsilon$, we have

$$\varepsilon_2^2 = 1 - \frac{b^2}{a^2}\sin^2\alpha - \frac{b^2}{c^2}\cos^2\alpha. \tag{7}$$

Employing the upper sign, and writing $\varepsilon_2'$ for $\varepsilon$, we have

$$\varepsilon_2'^2 = \frac{1 - \dfrac{b^2}{a^2}\sin^2\alpha - \dfrac{b^2}{c^2}\cos^2\alpha}{-\dfrac{b^2}{a^2}\sin^2\alpha - \dfrac{b^2}{c^2}\cos^2\alpha} = \frac{\varepsilon_2^2}{\varepsilon_2^2 - 1}. \tag{8}$$

One of the values, therefore, of $\varepsilon$ found by substituting (5) and (6) in (4) is the ordinary, and the other the conjugate, eccentricity.

Thus far we have made no assumption with regard to the relative lengths of the axes of the quadric. Let us suppose $a > b > c$; and put

$$\begin{aligned}
\frac{a^2 - b^2}{a^2} &= e_1^2, & \frac{b^2 - a^2}{b^2} &= e_1'^2, \\
\frac{a^2 - c^2}{a^2} &= e_2^2, & \frac{c^2 - a^2}{c^2} &= e_2'^2, \\
\frac{b^2 - c^2}{b^2} &= e_3^2, & \frac{c^2 - b^2}{c^2} &= e_3'^2;
\end{aligned} \tag{9}$$

in which the accents mark the conjugate eccentricities of the principal sections of the quadric. Substituting in (7), we find

$$\varepsilon_2^2 = e_1^2 \sin^2\alpha + e_3'^2 \cos^2\alpha. \tag{10}$$

Similarly, when $\gamma = 90°$,

$$\varepsilon_1^2 = 1 - \frac{c^2}{b^2}\sin^2\beta - \frac{c^2}{a^2}\cos^2\beta$$

$$= e_3^2 \sin^2\beta + e_2^2 \cos^2\beta, \tag{11}$$

$$\varepsilon_1'^2 = \frac{\varepsilon_1^2}{\varepsilon_1^2 - 1};$$

and when $\alpha = 90°$,

$$\varepsilon_3^2 = 1 - \frac{a^2}{c^2}\sin^2\gamma - \frac{a^2}{b^2}\cos^2\gamma$$

$$= e_3'^2 \sin^2\gamma + e_1'^2 \cos^2\gamma, \tag{12}$$

$$\varepsilon_3'^2 = \frac{\varepsilon_3^2}{\varepsilon_3^2 - 1}.$$

It is evident in each of these cases that the cutting plane is perpendicular to one of the principal planes of the quadric.

Thus far we have said nothing as to which of the values of $\varepsilon^2$ represent the ordinary and which the conjugate eccentricity. The choice of values depends upon the sign. In the case of the ellipse one of these values is always positive and the other always negative. In equation (11) $e_2^2$ and $e_3^2$ are both positive, and hence $\varepsilon_1^2$ is always positive and the square of the ordinary eccentricity. In (10) $e_1^2$ is positive and $e_3'^2$ is negative, and hence $\varepsilon_2^2$ will be sometimes positive and sometimes negative, depending on the value of $\alpha$. In (12) $e_1'^2$ and $e_2'^2$ are both negative, and hence $\varepsilon_3^2$ is always negative, and, in accordance with the above notation, must be written

$$\varepsilon_3'^2 = e_2'^2 \sin^2\gamma + e_1'^2 \cos^2\gamma.$$

Again, since $\sin^2\alpha = \cos^2\gamma$ when $\beta = 90°$, it is evident that (10) may be written

$$\varepsilon_2^2 = e_1^2 \cos^2\gamma + e_3'^2 \sin^2\gamma.$$

Similarly, (11) and (12) may be written

$$\varepsilon_1^2 = e_2^2 \cos^2\alpha + e_3^2 \sin^2\alpha,$$

$$\varepsilon_3'^2 = e_2'^2 \cos^2\beta + e_3'^2 \sin^2\beta.$$

Thus far we have restricted the investigation to the ellipsoid and found for $\varepsilon_1^2$, $\varepsilon_2^2$, $\varepsilon_3^2$ values which depend only upon a variable angle and the eccentricities of the principal sections. Hence in the above equations $e_1$, $e_2$, and $e_3$ are limited to values between 0 and 1. But if the ellipsoid by a variation of its axes becomes a paraboloid or hyperboloid, one or more of these quantities pass to unity or

beyond. Hence in order to extend the application of $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ to the general surface of the second degree we have only to consider $e_1$, $e_2$, and $e_3$ as capable of assuming all values between 0 and $\infty$.

## III.

### A SECOND METHOD OF FINDING $\varepsilon$.

There is another and indeed a much shorter method of finding $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, but it is applicable only when the cutting plane is perpendicular to one of the principal planes of the quadric, and not to the general case.

Take the ellipsoid whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and suppose a cutting plane perpendicular to the plane of $xy$ and passing through the centre. The equation of the principal section by the plane of $xy$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; or, in polar co-ordinates, the centre being the pole and the axis of $x$ the prime radius,

$$R_1^2 = \frac{a^2b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}. \tag{1}$$

Now $R_1$ is the semimajor axis of a section by a plane through the centre perpendicular to the plane of $xy$, and making an angle $\theta$ with the axis of $x$. $c$ is the semiminor axis of the section. Hence we have readily

$$\begin{aligned} \varepsilon_1^2 &= \frac{R_1^2 - c^2}{R_1^2} \\ &= 1 - \frac{c^2}{b^2}\sin^2\theta - \frac{c^2}{a^2}\cos^2\theta \\ &= e_2^2\cos^2\theta + e_3^2\sin^2\theta. \end{aligned} \tag{2}$$

In the same manner, we find in the plane of $xz$ that if

$$R_2^2 = \frac{a^2c^2}{a^2 \sin^2\theta + c^2 \cos^2\theta},$$

then

$$\begin{aligned} \varepsilon_2^2 &= 1 - \frac{b^2}{R_2^2} \\ &= 1 - \frac{b^2}{c^2}\sin^2\theta - \frac{b^2}{a^2}\cos^2\theta \\ &= e_1^2\cos^2\theta + e_3'^2\sin^2\theta. \end{aligned}$$

Since $R_2^2$ is sometimes greater and sometimes less than $b^2$, $\varepsilon_2^2$ is sometimes positive and sometimes negative.

In the $yz$ plane $a^2$ is always greater than the radius vector; whence, if

$$R_3^2 = \frac{b^2c^2}{b^2 \sin^2\theta + c^2 \cos^2\theta},$$

then

$$\varepsilon_3'^2 = 1 - \frac{a^2}{R_3^2}$$

$$= 1 - \frac{a^2}{c^2}\sin^2\theta - \frac{a^2}{b^2}\cos^2\theta$$

$$= e_1'^2 \cos^2\theta + e_3'^2 \sin^2\theta.$$

## IV.

### $\varepsilon_1$, $\varepsilon_2$, AND $\varepsilon_3$ FOR DIFFERENT VARIETIES OF QUADRICS.

The results obtained in the last section are the normal forms for the ellipsoid when $a^2 > b^2 > c^2$. If $c^2$ is negative, or both $b^2$ and $c^2$, we have the hyperboloid of one or of two sheets. The *form* of $\varepsilon_1^2$, $\varepsilon_2^2$, and $\varepsilon_3^2$ in either case remains unchanged. In the elliptic hyperboloid of one sheet, for which $c^2$ is negative, we find $e_1^2$ less than unity, and $e_2^2$ and $e_3^2$ both greater than unity. Since these quantities are independent of the absolute values of the axes, they still represent the eccentricities when the hyperboloid degenerates into its asymptotic cone. Hence, for such a cone the form of $\varepsilon_1^2$, $\varepsilon_2^2$, and $\varepsilon_3^2$ remains unchanged.

If $a^2 = \infty$, the other axes remaining the same, the surface is an elliptic paraboloid. $e_1^2$ and $e_2^2$ are both equal to unity, and hence $e_1'^2$ and $e_2'^2$ are both equal to infinity.

If $b^2 = c^2$, the surface is one of revolution around the axis of $x$. $e_1^2 = e_2^2$ and $e_3^2 = 0$. Hence (2), (3), and (4) of last section become

$$\varepsilon_1^2 = \varepsilon_2^2 = e_1^2 \cos^2\theta \quad \text{and} \quad \varepsilon_3'^2 = e_1'^2. \tag{1}$$

If $a^2 = b^2$, the surface is one of revolution around the axis of $z$, and we have

$$\varepsilon_2^2 = \varepsilon_3^2 = e_3'^2 \sin^2\theta \quad \text{and} \quad \varepsilon_1^2 = e_3^2. \tag{2}$$

If in this case $c^2$ is greater than $a^2$, $e_3'^2$ is of course real. This last formula remains true when the hyperboloid of revolution degenerates into its asymptotic cone, which is a right circular cone.

The meaning of $e_3'^2$ in this case deserves special consideration. $e_3'^2 = \dfrac{c^2 + b^2}{c^2}$, where $c$ represents a certain distance from the apex of the cone measured on the axis of $z$. We may so choose the base of the cone that $c$ shall represent the distance from the apex to the centre of the base. Then $b = a$ is the radius of

the base. Let $\varphi$ be the angle which any element of the cone makes with the plane of the base. We have then a right triangle in which $c^2 + b^2$ is the square on the hypothenuse and $c^2$ the square on the side opposite $\varphi$. Hence

$$\frac{c^2 + b^2}{c^2} = e_3'^2 = \frac{1}{\sin^2\varphi};$$

$$\therefore \quad \varepsilon^2 = \frac{\sin^2\theta}{\sin^2\varphi}. \qquad (3)$$

If $\varphi = 90°$, the cone becomes a cylinder and $\varepsilon^2 = \sin^2\theta$. This last result follows immediately from the geometry of the cylinder, since the semiminor axis of a section is always equal to the radius of the cylinder, and the semimajor axis is $b \sec \theta$. Hence

$$\varepsilon^2 = \frac{b^2 \sec^2\theta - b^2}{b^2 \sec^2\theta} = \sin^2\theta.$$

## V.

### A THIRD METHOD OF FINDING $\varepsilon$.

There is a third method of finding $\varepsilon$, which may be applied separately to each variety of quadric. I will apply it to the case of the right circular cone.

The equation of a cone, asymptotic to an hyperboloid of revolution, the origin being at the vertex, is

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 0. \qquad (1)$$

To transform this to a new origin on the axis of $z$ at a distance $d$ below the vertex, we write $z - d$ for $z$; whence

$$(x^2 + y^2)\frac{c^2}{a^2} = (z - d)^2. \qquad (2)$$

Now $\frac{c}{a}$ is the tangent of the angle which an element of the cone makes with the base. Calling this angle $\varphi$, we have

$$(x^2 + y^2)\tan^2\varphi = (z - d)^2. \qquad (3)$$

Let the cone be cut by any plane

$$z = x \tan\theta, \qquad (4)$$

passing through the axis of $y$, where $\theta$ is the angle which the cutting plane makes with the base. Eliminating $z$ between (3) and (4), we have

$$(x^2+y^2)\tan^2\varphi = (x \tan\theta - d)^2, \tag{5}$$

which is the equation of the projection of the curve of intersection on the plane of $xy$. The plane of the curve meets the plane of $xy$ in the axis of $y$. To obtain the equation of the curve itself we have only to replace $x$ by $x\cos\theta$. Substituting and expanding,*

$$x^2(\tan^2\varphi - \tan^2\theta)\cos^2\theta + y^2\tan^2\varphi + 2dx\sin\theta = d^2, \tag{6}$$

which is the equation of a conic. Since this equation contains no term in $xy$, we know that the coefficients of $x^2$ and $y^2$ are the squares of the semiaxes. Substituting these coefficients in the formula for $\varepsilon^2$, we have

$$\varepsilon^2 = \frac{a^2-b^2}{a^2} = \frac{\sin^2\theta}{\sin^2\varphi}. \tag{7}$$

When $\theta$ and $\varphi$ both lie between 0° and 90° it is evident that $\varepsilon$ is less, equal to, or greater than unity according as $\theta$ is less, equal to, or greater than $\varphi$; and the curve is respectively an ellipse, parabola, or hyperbola. $\varepsilon = 0$ when $\theta = 0$, and the curve is a circle.

By a process exactly similar to the above we might deduce the expression for $\varepsilon$ for every variety of surface of second degree.

The foregoing theory of the eccentricity of plane sections of quadrics is not only interesting in itself and capable of yielding new results, but leads by direct and easy paths to many old theorems which have hitherto been reached only by roundabout processes.

*This equation is given in many elementary text-books. See Wentworth, Ray, etc.